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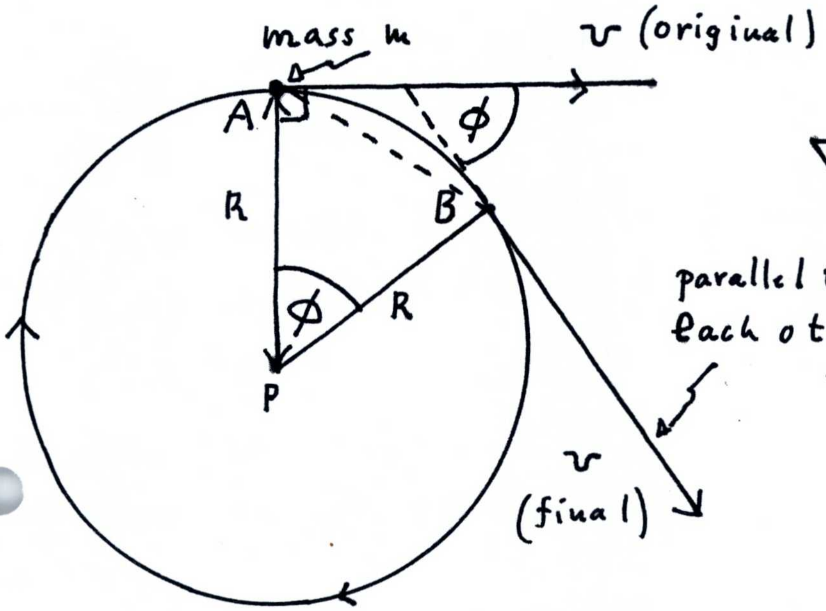
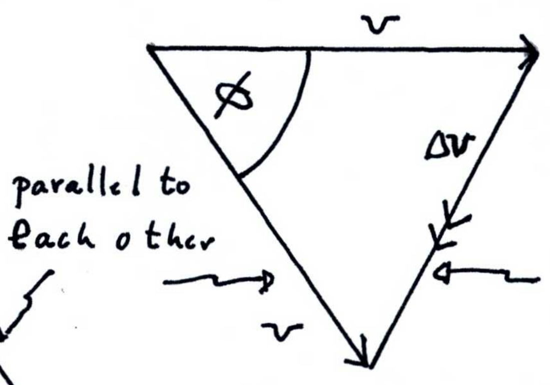
Deriving an expression for the centripetal acceleration of a secondary body revolving around a primary body



Subtracting the two vectors: tail-to-tail



This is the change in velocity, Δv, of the orbiting secondary body, during a time interval of Δt.

Note that the magnitude of each of the two vectors is the same as that of the other, because the speed of the orbiting secondary does not change. φ has necessarily been exaggerated, so that the diagram is clear.

Consider the triangle (chord) (A B)P

This triangle has two sides equal in length (= radius of the orbiting secondary), R, with angle φ as shown.

Ideally, I would have drawn $\phi \ll 1^\circ$, so that Δt the time interval became vanishingly small. Try to imagine such a small interval of time. For so small an angle φ, the arc AB is close in length to the chord AB.

Consider the triangle portrayed in the right-hand diagram. This also has two sides equal in length, v. (the magnitude of the velocity vectors), with the same angle, φ.

We see that these triangles are similar

$\Longrightarrow$

$$\frac{AB}{R} = \frac{\Delta v}{v} \longrightarrow (1)$$

Rearranging :

$$\Delta v = \frac{AB.\,v}{R} \longrightarrow (2)$$

change in velocity, remember ?

We postulated that the orbiting secondary moves from A to B during a time interval of Δt.

If the left-hand side of equation (2) is divided by Δt, this will give us the <u>rate of change of velocity</u>

That is, <u>acceleration</u>.

Naturally, the right-hand side must also be divided by Δt

$$\therefore \vec{a}_{\text{centripetal}} = \frac{AB.\,v}{R.\,\Delta t} \longrightarrow (3)$$

Now, look closely at the right-hand side of equation (3)

$$\frac{AB}{\Delta t} = \frac{\text{distance travelled by the secondary}}{\text{time}}$$

That is, velocity, v.

The numerator of the right-hand side of equation (3) now contains

$$v \times v = v^2$$

$$\therefore \vec{a}_{\text{centripetal}} = \frac{v^2}{R} \longrightarrow (4)$$

v has units of $m\ s^{-1}$

$\therefore v^2$ has units $(m\ s^{-1})^2$

$= m^2\ s^{-2}$

$\therefore$ the right-hand side of equation (4) has units $\frac{m\ s^{-2}}{m}$, the units of acceleration.

Since the mass of the orbiting secondary is m, it follows that the centripetal force required to maintain circular motion is

$[\text{Using } \vec{F} = m\vec{a}]$

$m \times$ centripetal acceleration

$$\therefore \boxed{\vec{F}_{\text{Centripetal}} = m.\frac{v^2}{R}} \longrightarrow (5)$$

Suppose the orbiting secondary completes one revolution during a time interval T.

$\longrightarrow$

This could, perhaps, be the sidereal period of a planet (secondary) orbiting the Sun (primary)

$\therefore$ the circumference of the planetary orbit will be $2\pi R$

$\therefore$ its orbital speed will be: $\dfrac{\text{total distance travelled}}{\text{total time taken}}$

$$= \frac{2\pi R}{T}$$

Now substitute for v, and then v^2, into equation 5

$$\vec{F}_{\text{centripetal}} = m \cdot \frac{\left(\frac{2\pi R}{T}\right)^2}{R}$$

$$\therefore \vec{F}_{\text{centripetal}} = m \cdot \frac{4\pi^2 R^2}{T^2 R}$$

Which reduces to:

$$\boxed{\vec{F}_{\text{centripetal}} = m \cdot \frac{4\pi^2 R}{T^2}} \quad \text{—— (6)}$$

Check the units of the right-hand side of the "boxed" equation above.

So, this centripetal force is provided by gravitation.

From earlier work

$$\vec{F}_{\text{gravitational}} = \frac{G M m}{R^2 \text{ (or } d^2)}$$

$\therefore$

$$\frac{GMm}{R^2} = m\frac{v^2}{R}$$

Dividing throughout by m:

$$\frac{GM}{R^2} = \frac{v^2}{R} \quad \text{—— (7)}$$

From equation 6, replace $\dfrac{4\pi^2 R}{T^2}$ into equation 7

$$\frac{GM}{R^2} = \frac{4\pi^2 R}{T^2}$$

Multiply throughout by R^2:

$$GM = \frac{4\pi^2 R^3}{T^2}$$

Divide throughout by $4\pi^2$:

$$\frac{GM}{4\pi^2} = \frac{R^3}{T^2}$$

or

$$\boxed{\frac{T^2}{R^3} = \frac{4\pi^2}{GM}}$$

$M = M_{\odot}$

Kepler's Third Law of Planetary Motion.

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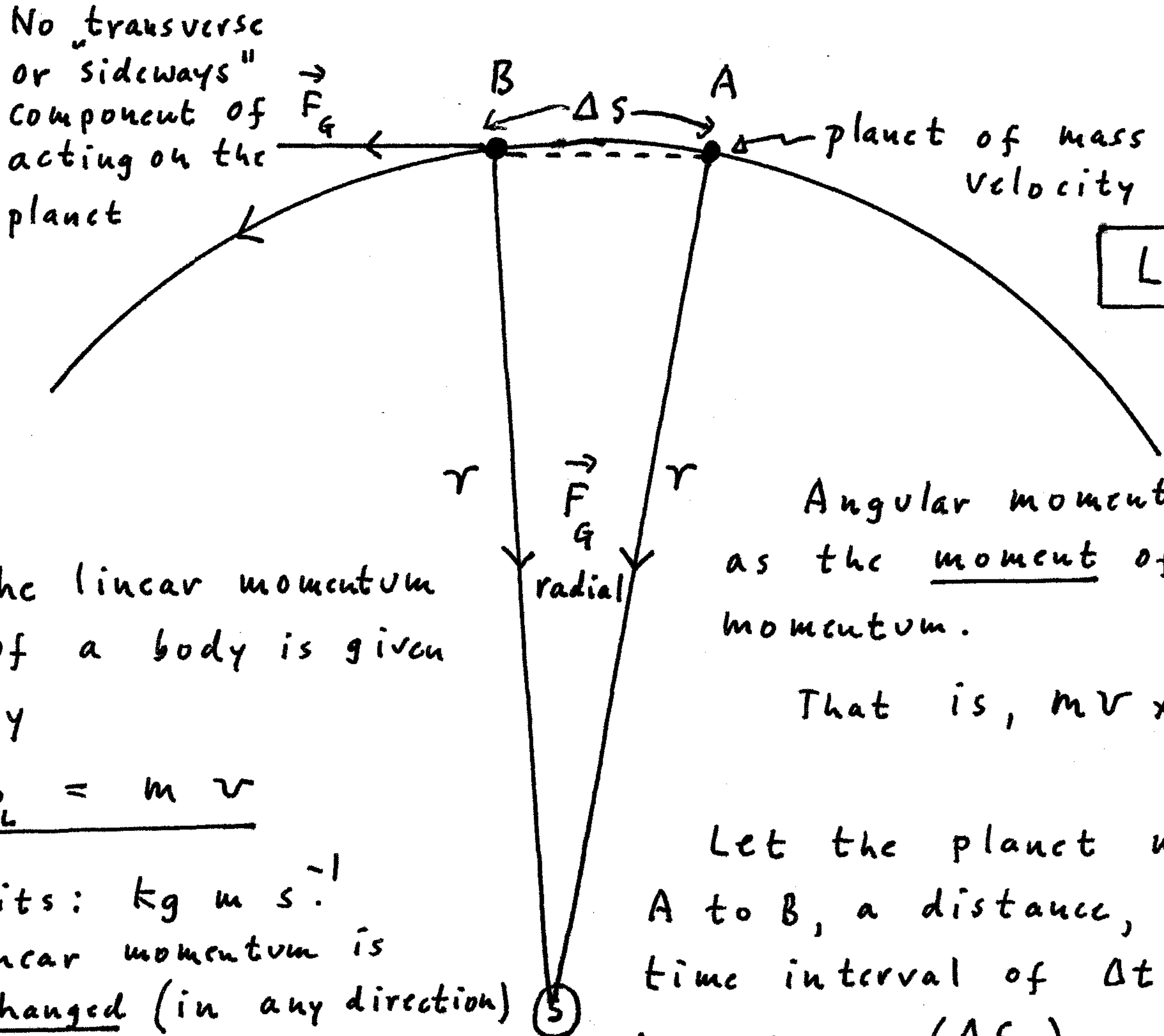
Kepler's Law of Equal Areas and the Conservation of Angular Momentum



$$\boxed{L_A = m v r}$$

The linear momentum of a body is given by

$$\underline{p_L = m v}$$

units: kg m s^{-1}

Linear momentum is unchanged (in any direction) unless there is an external force acting.

Angular momentum is defined as the moment of the linear momentum.

That is, $mv \times \begin{pmatrix}\text{orbital radius} \\ \text{of the planet}\end{pmatrix}$

Let the planet move from A to B, a distance, Δs, during a time interval of Δt.

$$\therefore \quad v = \left(\frac{\Delta s}{\Delta t}\right)$$

In the "boxed" expression, upper right, replace v by $\left(\frac{\Delta s}{\Delta t}\right)$

Hence,

$$\boxed{L_A = m\left(\frac{\Delta s}{\Delta t}\right) r}$$

Units: kg m s^{-1} × m = kg m^2 s^{-1}

Since the force acting on the planet is entirely radial, that is, there is no transverse component, it follows that the angular momentum of the planet must be constant

so, $m\left(\frac{\Delta s}{\Delta t}\right) r = \text{constant}$

Consider the triangle ABS. Its area $= \frac{1}{2}$ base × height

$$= \frac{1}{2} \Delta s \times r = \Delta A$$

$$\therefore \quad \Delta s \times r = 2\Delta A$$

$$\therefore \quad m \cdot 2\left(\frac{\Delta A}{\Delta t}\right) = \text{constant}$$

$$\Rightarrow \boxed{\frac{\Delta A}{\Delta t} = \frac{\text{constant}}{2m}}$$

D. F. 2017, Sept. 21